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Describing Analytically the Matter-Enhanced Two-Neutrino Transitions in a Medium

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Abstract

A general exact analytic expression for the probability of matter-enhanced two-neutrino transitions in a medium (MSW, RSFP, generated by neutrino FCNC interactions, etc.) is derived. The probability is expressed in terms of three real functions of the parameters of the transitions: the “jump” probability and two phases (angles). The results obtained can be utilized, in particular, in the studies of the matter-enhanced transitions/conversions of solar and supernova neutrinos. An interesting similarity between the Schroedinger equation for the radial part of the non-relativistic wave function of the hydrogen atom and the equation governing the MSW transitions of solar neutrinos in the exponentially varying matter density in the Sun is also briefly discussed.

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1. Introduction

In the present article we derive exact, simple and general analytic expression for the probability of matter-enhanced two-neutrino transitions in a medium. The matter-enhanced neutrino transitions provide, as is well-known, at least three different varieties of neutrino physics solutions of the solar neutrino problem [1–5]. Such transitions, in particular, can play important role in the supernovae dynamics [6,7] and can be at the origin of the observed large space velocities of pulsars [8]. Even in the simplest case of transitions involving only two weak-eigenstate neutrinos, however, the system of evolution equations describing the transitions does not admit, in general, exact solutions and one has to rely on numerical methods to calculate the corresponding transition probabilities. There are few notable exceptions: the evolution equations can be solved exactly, for instance, in the case of MSW transitions [1,2] in matter with density which changes linearly [9–11] (see also [12]) or exponentially [13,14] along the neutrino path. On the basis of the exact solutions expressed in the two cases respectively in terms of parabolic cylinder and confluent hypergeometric functions, simple expressions for the neutrino transition probabilities, containing only elementary functions, have been obtained¹ [9–11,13]. The case of exponentially varying density is especially relevant for the analytic description of the solar neutrino transitions in the Sun since according to the contemporary solar models [17] the density decreases approximately exponentially from the center to the surface of the Sun. Indeed, it was found that the expression for the average probability obtained in the exponential density approximation provides a very precise (and actually, the most precise) description of the MSW transitions of the solar neutrinos in the Sun [18].

The expression for the average probability derived in the linear density approximation contains as an integral part the result by Landau and Zener for the corresponding “jump” probability [12,16]. It is widely used, for example, in the studies of the MSW transitions and resonance spin-flavour precession (RSFP) or conversion [3] of the supernova neutrinos (see, e.g., [6–8]). However, its accuracy in describing the supernova neutrino transitions/conversions has never been tested. More specifically, the oscillating terms present in the relevant transition/conversion probabilities have always been neglected without any control on the validity of this approximation because no explicit expressions for these terms could be derived².

The analytical result for the two-neutrino transition/conversion probability we obtain in the present study, in addition of being exact and rather simple, is also general and universal in form. It is valid for two-neutrino MSW transitions, RSFP, for matter- enhanced transitions

¹The expression for the average MSW neutrino transition probability in the case of linearly varying density was derived first in ref. [15] using qualitative arguments and the Landau - Zener formula [12,16] for the “jump” probability.

²One exception is the first article in ref. [7] where the results of the present study have been used to ensure that the corresponding oscillating terms are strongly suppressed.

induced by neutrino flavour-changing neutral current (FCNC) interactions [4,5], etc. In all these cases the neutrino transition/conversion probabilities are shown to be given by expressions which have one and the same universal structure. Such an expression was derived first in ref. [19] for the solar neutrino MSW transition probability in the exponential density approximation.

2. The Probability of Two-Neutrino Transitions in a Medium: General Results

Consider neutrino transitions in a medium, which involve two weak-eigenstate neutrinos ν_α and ν_β , $\nu_\alpha \rightarrow \nu_\beta$, $\nu_\alpha \neq \nu_\beta = \overset{(-)}{\nu}_e, \overset{(-)}{\nu}_\mu, \overset{(-)}{\nu}_\tau, \overset{(-)}{\nu}_s$, ν_s being a sterile neutrino ³. We shall assume that the transitions are described by a system of evolution equations which has (or can be reduced to) the form:

$$i \frac{d}{dt} \begin{pmatrix} A_\alpha(t, t_0) \\ A_\beta(t, t_0) \end{pmatrix} = \begin{pmatrix} -\epsilon(t) & \epsilon'(t) \\ \epsilon'(t) & \epsilon(t) \end{pmatrix} \begin{pmatrix} A_\alpha(t, t_0) \\ A_\beta(t, t_0) \end{pmatrix} \quad (1)$$

where $A_\alpha(t, t_0)$ ($A_\beta(t, t_0)$) is the amplitude of the probability to find neutrino ν_α (ν_β) at time t of the evolution of the neutrino system if at time t_0 the neutrino ν_α or ν_β has been produced, $t \geq t_0$, $\epsilon(t)$ and $\epsilon'(t)$ are real functions of some of the physical quantities characterizing the medium (density, magnetic field strength, etc.) and the neutrino system in vacuum (neutrino energy, mass squared difference, etc.), and we have omitted the indices α and β from $\epsilon(t)$ and $\epsilon'(t)$. In writing the evolution equations in the form (1) we have supposed that the effects of neutrino absorption, creation and of possible neutrino instability (or loss of coherence) in the evolution of the neutrino system are negligible. This corresponds to a large number of physically interesting cases, in particular, those discussed in [1–8]. Under the above assumption the evolution matrix of the system is hermitian and the equations describing the two-neutrino transitions in a medium can always be brought (by phase transformations of the two probability amplitudes) to the form (1) with real $\epsilon(t)$ and $\epsilon'(t)$.

There are two specific types of initial conditions for the system (1) relevant to our discussion:

$$A_\alpha^0 = 1, \quad A_\beta^0 = 0, \quad (A)$$

and

$$A_\alpha^0 = 0, \quad A_\beta^0 = 1, \quad (B)$$

where $A_\alpha^0 = A_\alpha(t_0, t_0)$ and $A_\beta^0 = A_\beta(t_0, t_0)$. If the initial conditions (A) hold, $A_\alpha(t, t_0)$ and $A_\beta(t, t_0)$ are probability amplitudes of the ν_α survival and of the $\nu_\alpha \rightarrow \nu_\beta$ transition while the neutrinos propagate from the point of their production to the point of neutrino trajectory reached at time t : $A_\alpha(t, t_0) = A(\nu_\alpha \rightarrow \nu_\alpha)$ and $A_\beta(t, t_0) = A(\nu_\alpha \rightarrow \nu_\beta)$ (we have not indicated explicitly that $A(\nu_\alpha \rightarrow \nu_\alpha)$ and $A(\nu_\alpha \rightarrow \nu_\beta)$ depend on t and t_0). Similarly, if the initial conditions (B) are valid we can write $A_\alpha(t, t_0) = A(\nu_\beta \rightarrow \nu_\alpha)$, $A_\beta(t, t_0) = A(\nu_\beta \rightarrow \nu_\beta)$,

³In what follows the term “transition” will be used to denote the process $\nu_\alpha \rightarrow \nu_\beta$ of the change of the type of the weak-eigenstate neutrino for all possible different cases, including the case of RSFP.

in accordance with the interpretation of the probability amplitudes $A_\alpha(t, t_0)$ and $A_\beta(t, t_0)$. In the case of most general initial conditions

$$A_\alpha^0 \neq 0, A_\beta^0 \neq 0, |A_\alpha^0|^2 + |A_\beta^0|^2 = 1, \quad (2)$$

the solutions of the system (1) are expressed as linear combinations of the solutions corresponding to the initial conditions (A) and (B):

$$A_\alpha(t, t_0) = A_\alpha^0 A(\nu_\alpha \rightarrow \nu_\alpha) + A_\beta^0 A(\nu_\beta \rightarrow \nu_\alpha), \quad (3a)$$

$$A_\beta(t, t_0) = A_\alpha^0 A(\nu_\alpha \rightarrow \nu_\beta) + A_\beta^0 A(\nu_\beta \rightarrow \nu_\beta). \quad (3b)$$

Evidently, for any of the initial conditions (A), (B) or (2) one has in the case under study

$$|A_\alpha(t, t_0)|^2 + |A_\beta(t, t_0)|^2 = 1, \quad (4)$$

which implies

$$A(\nu_\alpha \rightarrow \nu_\alpha) A^*(\nu_\beta \rightarrow \nu_\alpha) = -A^*(\nu_\beta \rightarrow \nu_\beta) A(\nu_\beta \rightarrow \nu_\alpha). \quad (5)$$

We give next several concrete examples of matter-enhanced neutrino transitions in a medium to which our general results will apply. In the case of MSW $\nu_e \rightarrow \nu_{\mu(\tau)}$ transitions of solar or supernova neutrinos we have [1,20]

$$\epsilon(t) = \frac{1}{2} \left[\frac{\Delta m^2}{2E} \cos 2\theta - \sqrt{2} G_F N_e(t) \right], \quad \epsilon'(t) = \frac{\Delta m^2}{4E} \sin 2\theta, \quad (6)$$

where $\Delta m^2 = m_2^2 - m_1^2$, $m_{1,2}$ being the masses of two neutrinos $\nu_{1,2}$ with definite mass in vacuum, E is the neutrino energy, θ is the neutrino mixing angle in vacuum, and $N_e(t)$ is the electron number density at the point of neutrino trajectory in the Sun or supernova, reached at time t . For the transitions $\nu_{\mu(\tau)} \rightarrow \nu_e$ which can take place in supernovae, $\epsilon(t)$ and $\epsilon'(t)$ are also given by (6). If the MSW transitions are into sterile neutrino, $\nu_e \rightarrow \nu_s$, $N_e(t)$ in eq. (6) has to be replaced by [21] $(N_e(t) - \frac{1}{2}N_n(t))$, where $N_n(t)$ is the neutron number density. In supernovae MSW $\nu_{\mu(\tau)} \rightarrow \nu_s$ transitions can also take place. In the latter case $N_e(t)$ in eq. (6) has to be substituted by $(-\frac{1}{2}N_n(t))$. Finally, for the corresponding transitions involving antineutrinos the term with the Fermi constant in $\epsilon(t)$ changes sign.

Another relevant example is the simplest version of neutrino resonance spin-flavour precession, e.g., $\nu_e \rightarrow \bar{\nu}_{\mu(\tau)}$, in the Sun or in supernovae. It is described by (1) with [3]

$$\epsilon(t) = \frac{1}{2} \left[\frac{\Delta m^2}{2E} - \sqrt{2} G_F (N_e(t) - N_n(t)) \right], \quad \epsilon'(t) = \mu_\nu B_\perp(t). \quad (7)$$

In eq. (7) $\Delta m^2 = m_2^2 - m_1^2 \equiv m^2(\bar{\nu}_{\mu(\tau)}) - m^2(\nu_e)$, μ_ν is a $\nu_e - \bar{\nu}_{\mu(\tau)}$ transition magnetic moment of a dipole type and $B_\perp(t)$ is the value of the component of the solar or supernova magnetic field perpendicular to the neutrino momentum. The RSFP of the type $\bar{\nu}_e \rightarrow \nu_{\mu(\tau)}$, which was shown in ref. [7] to be one of the possible mechanisms of supernova shock revival, is also described by (1) with $\epsilon(t)$ and $\epsilon'(t)$ given by eq. (7) in which the term containing the Fermi constant has an opposite sign. For neutrino RSFP in a twisting magnetic field [22,23] as well as MSW transitions or RSFP of supernova neutrinos, which can be responsible for

the observed large space velocities of pulsars [8], the expressions for $\epsilon'(t)$ coincide with those given in eqs. (6) (MSW transitions) and (7) (RSFP), while the expressions (6) and (7) for $\epsilon(t)$ are modified (but not essentially from the point of view of the problem we are interested in) by the presence of additional terms ⁴.

Finally, the system (1) with

$$\epsilon(t) = \frac{1}{2}\sqrt{2}G_F N_e(t) [\lambda(1 + 2 \frac{N_n(t)}{N_e(t)}) - 1], \quad \epsilon'(t) = \sqrt{2}G_F N_e(t) \lambda'(1 + 2 \frac{N_n(t)}{N_e(t)}), \quad (8)$$

where λ and λ' are real constants, $\lambda > 0$, corresponds to matter-enhanced transitions of solar ($\nu_e \rightarrow \nu_\tau$) or supernova ($\overset{(-)}{\nu}_e \rightarrow \overset{(-)}{\nu}_{\tau(\mu)}$, $\overset{(-)}{\nu}_{\tau(\mu)} \rightarrow \overset{(-)}{\nu}_e$) neutrinos induced by ν_e flavour-changing and new ν_e and $\nu_{\tau(\mu)}$ flavour-conserving but flavour non-symmetric, neutral current interactions (on the d -quark) [4]. Note that in the latter case matter-enhanced $\overset{(-)}{\nu}_e \rightarrow \overset{(-)}{\nu}_{\tau(\mu)}$ ($\overset{(-)}{\nu}_{\tau(\mu)} \rightarrow \overset{(-)}{\nu}_e$) transitions are possible even if neutrinos have zero mass and neutrino mixing is absent in vacuum.

Our aim is to derive an exact and general expression, e.g., for the ν_α survival probability assuming that the functions $\epsilon(t)$ and $\epsilon'(t)$ are given ⁵. This is typically the case in the studies of the effects of the matter-enhanced transitions of the solar and supernova neutrinos, indicated above. In these studies one uses the predictions for the electron and neutron number density distributions and for the neutrino energy spectra provided by the standard solar models and by the models of the supernovae. The energy spectrum of solar neutrinos is practically solar model independent and is known with a relatively high precision [17]. The supernova neutrino energy spectra are somewhat model dependent [6,7]. No direct information exists about the magnetic fields in the solar and supernovae interiors, and in the studies of the neutrino conversion in which they are presumed to play important role, one uses plausible field configurations (see, e.g., the third article quoted in ref. [3] and ref. [7]). The magnetic field strength and Δm^2 in the case of RSFP, the neutrino mixing angle in vacuum θ and Δm^2 in the case of MSW transitions, and the constants λ and λ' in the example specified by eq. (8), are often treated as free parameters to be determined by the physics of the problem being investigated. We will not specify the form of $\epsilon(t)$ and $\epsilon'(t)$ in our further analysis. Our results will be valid for any $\epsilon(t)$ and $\epsilon'(t)$ and, in particular, for $\epsilon(t)$ and $\epsilon'(t)$ given in eqs. (6) - (8) and the cases discussed above.

Let us remind the reader that neutrinos go through a resonance point reached at time t_{res} , $t_0 < t_{res} < t$, on their way to the final point of their trajectory if the function $\epsilon(t)$ changes sign going through zero at $t = t_{res}$, $\epsilon(t_{res}) = 0$, while $\epsilon'(t_{res}) \neq 0$ (see the first article quoted in ref. [20] as well as ref. [2] and, e.g., [3,4,24]). These conditions are typically fulfilled in most of the cases of physical interest, in which also one has $\epsilon'(t) \neq 0$ on the whole neutrino trajectory, except possibly at the trajectory's final point. For $\nu_e \rightarrow \nu_{\mu(\tau)}$ MSW

⁴In all the indicated cases the neutrinos are assumed to be relativistic.

⁵The results we shall obtain can be relevant also for the inverse problem, namely, the problem of reconstructing $\epsilon(t)$ and/or $\epsilon'(t)$ from data about ν_α ($\nu_\alpha \rightarrow \nu_\beta$) survival (transition) probability.

transitions, for instance, the resonance condition can be satisfied, as it follows from eq. (6), if, e.g., $\Delta m^2 > 0$ and $\cos 2\theta > 0$, and these two inequalities will be assumed to hold in our further discussion of the MSW effect for solar neutrinos.

It is convenient to introduce the neutrino mixing angle in matter,

$$\sin 2\theta_m(t) = \frac{\epsilon'(t)}{\sqrt{\epsilon^2(t) + \epsilon'^2(t)}}, \quad (9)$$

and the neutrino matter-eigenstates at time t , $|\nu_{1,2}^m(t)\rangle$:

$$|\nu_\alpha\rangle = |\nu_1^m(t)\rangle \cos \theta_m(t) + |\nu_2^m(t)\rangle \sin \theta_m(t), \quad (10a)$$

$$|\nu_\beta\rangle = -|\nu_1^m(t)\rangle \sin \theta_m(t) + |\nu_2^m(t)\rangle \cos \theta_m(t). \quad (10b)$$

At the resonance point we have: $\sin^2 2\theta_m(t_{res}) = 1$. The states $|\nu_{1,2}^m(t)\rangle$, which are also called “adiabatic”, are the instantaneous eigenstates of the evolution matrix (Hamiltonian) in (1) at time t , corresponding to the two eigenvalues, $E_{1,2}^m(t)$, whose difference is given by

$$E_2^m - E_1^m = 2\sqrt{\epsilon^2(t) + \epsilon'^2(t)}. \quad (11)$$

Under certain specific conditions (e.g., ν_e produced in the Sun at densities $N_e(t_0)$ much larger than the resonance density $N_e^{res} = \Delta m^2 \cos 2\theta / (2E\sqrt{2}G_F)$) the states $|\nu_{1,2}^m(t_0)\rangle$, for example, can practically coincide with the states $|\nu_{\beta,\alpha}\rangle$.

We shall denote by $A(\nu_i^m(t_0) \rightarrow \nu_j^m(t)) \equiv A_{ij}^m(t, t_0)$ the probability amplitudes of the $\nu_i^m(t_0) \rightarrow \nu_j^m(t)$ transitions, $i, j = 1, 2$, which take place when the neutrinos propagate in the medium. The “jump” (or “level crossing”) probability, P' ,

$$P' = |A(\nu_1^m(t_0) \rightarrow \nu_2^m(t))|^2 = |A_{12}^m|^2, \quad (12)$$

plays a very important role in the neutrino matter-enhanced transitions in a medium: its value determines the type of the $\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)}$ transition (for $P' \cong 0$ it is adiabatic and nonadiabatic otherwise) and typically controls the value of the transition probability in a large region of the corresponding parameter space⁶. Moreover, in a wide class of cases one can use the Landau-Zener result for P' [12,16] or, e.g., its analog derived in the exponential density approximation [13]. Then P' is determined by the values of $\epsilon(t)$, $\epsilon'(t)$ and of their derivatives at the resonance point (see further). Therefore we would like to find an exact general expression for the $\nu_{\alpha(\beta)}$ survival and $\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)}$ transition probabilities in terms of the “jump” probability. To this end let us express the four probability amplitudes $A(\nu_\alpha \rightarrow \nu_{\alpha(\beta)})$ and $A(\nu_\beta \rightarrow \nu_{\alpha(\beta)})$ in terms of the amplitudes A_{ij}^m , $i, j = 1, 2$. This can be done using the relations (10a) and (10b):

$$A(\nu_\alpha \rightarrow \nu_\alpha) = A_{11}^m c_m^0 c_m + A_{12}^m c_m^0 s_m + A_{21}^m s_m^0 c_m + A_{22}^m s_m^0 s_m, \quad (13a)$$

⁶As we shall see later, one has in the case of interest: $|A(\nu_2^m(t_0) \rightarrow \nu_1^m(t))|^2 = |A(\nu_1^m(t_0) \rightarrow \nu_2^m(t))|^2 = P'$.

$$A(\nu_\alpha \rightarrow \nu_\beta) = -A_{11}^m c_m^0 s_m + A_{12}^m c_m^0 c_m - A_{21}^m s_m^0 s_m + A_{22}^m s_m^0 c_m, \quad (13b)$$

$$A(\nu_\beta \rightarrow \nu_\alpha) = -A_{11}^m s_m^0 c_m - A_{12}^m s_m^0 s_m + A_{21}^m c_m^0 c_m + A_{22}^m c_m^0 s_m, \quad (13c)$$

$$A(\nu_\beta \rightarrow \nu_\beta) = A_{11}^m s_m^0 s_m - A_{12}^m s_m^0 c_m - A_{21}^m c_m^0 s_m + A_{22}^m c_m^0 c_m, \quad (13d)$$

where $c_m^{(0)} = \cos \theta_m(t_{(0)})$ and $s_m^{(0)} = \sin \theta_m(t_{(0)})$.

For any fixed t_0 and $t \geq t_0$ one has from (4) and (13a) - (13d):

$$|A_{11}^m|^2 + |A_{12}^m|^2 = 1, \quad |A_{21}^m|^2 + |A_{22}^m|^2 = 1, \quad A_{11}^m (A_{21}^m)^* = -(A_{22}^m)^* A_{12}^m. \quad (14)$$

We shall prove next that the solutions of the system of equations (1) corresponding to the initial conditions (A) and (B) satisfy the following relations:

$$A^*(\nu_\alpha \rightarrow \nu_\alpha) = A(\nu_\beta \rightarrow \nu_\beta), \quad A(\nu_\beta \rightarrow \nu_\alpha) = -A^*(\nu_\alpha \rightarrow \nu_\beta). \quad (15)$$

Indeed ⁷, using the first equation in (1) to express $A_\beta(t, t_0)$ as

$$A_\beta(t, t_0) = \frac{1}{\epsilon'} (\epsilon + i \frac{d}{dt}) A_\alpha(t, t_0), \quad (16)$$

and substituting eq. (16) in the second equation in (1) we get a second order differential equation for $A_\alpha(t, t_0)$:

$$\left\{ \frac{d^2}{dt^2} - \frac{\dot{\epsilon}'}{\epsilon'} \frac{d}{dt} + [\epsilon^2 + \epsilon'^2 - i\epsilon (\frac{\dot{\epsilon}}{\epsilon} - \frac{\dot{\epsilon}'}{\epsilon'})] \right\} A_\alpha(t, t_0) = 0, \quad (17)$$

where $\dot{\epsilon}' = \frac{d}{dt} \epsilon'$. It is easy to show that $A_\beta^*(t, t_0)$ satisfies exactly the same second order differential equation: this can be checked, e.g., by taking the complex conjugate of the two equations in (1) and by using the second equation written as

$$-A_\alpha^*(t, t_0) = \frac{1}{\epsilon'} (\epsilon + i \frac{d}{dt}) A_\beta^*(t, t_0), \quad (18)$$

to eliminate $(-A_\alpha^*(t, t_0))$ from the first equation. Moreover, the initial conditions (A) lead to initial conditions for the solution $A_\alpha(t, t_0)$ of the equation (17), which coincide with the initial conditions for the solution $A_\beta^*(t, t_0)$, following from conditions (B). This fact and the relations (16) and (18) lead to ⁸ eq. (15).

It is easy to convince oneself utilizing eqs. (13a) - (13d) and (15) that the amplitudes A_{ij}^m satisfy analogous relations:

$$A_{22}^m = (A_{11}^m)^*, \quad A_{21}^m = -(A_{12}^m)^*. \quad (19)$$

⁷For the special case of MSW $\nu_e \rightarrow \nu_{\mu(\tau)}$ transitions, eq. (6), these relations were shown to be valid in refs. [13,25].

⁸Obviously, the proof of eq. (15) presented here relies on the assumption that $\epsilon'(t) \neq 0$ and that the derivatives of $\epsilon(t)$ and $\epsilon'(t)$ exist on the neutrino trajectory.

These relations can be derived also from the systems of evolution equations for the amplitudes A_{11}^m , A_{12}^m and for the amplitudes A_{21}^m , A_{22}^m which can be obtained from (1) by using (13a) - (13d). It follows from (19), in particular, that:

$$\Phi_{11}(t, t_0) = -\Phi_{22}(t, t_0), \quad \Phi_{21}(t, t_0) = \pi - \Phi_{12}(t, t_0), \quad (20)$$

where $\Phi_{ij}(t, t_0)$ is the phase of the amplitude $A_{ij}^m(t, t_0)$,

$$\arg(A_{ij}^m) = \Phi_{ij}, \quad i, j = 1, 2. \quad (21)$$

Thus, eqs. (14) and (19) imply that the four complex amplitudes A_{ij}^m , $i, j = 1, 2$, depend actually only on three real functions of the parameters of the problem, which can be chosen to be the “jump” probability $P' = |A_{12}^m|^2 = |A_{21}^m|^2$ and the phase functions $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$.

It is not difficult to find now the expression of interest for the ν_α survival probability, $P(\nu_\alpha \rightarrow \nu_\alpha)$, assuming the initial conditions (A) are valid: $P(\nu_\alpha \rightarrow \nu_\alpha) = |A(\nu_\alpha \rightarrow \nu_\alpha)|^2$. Note that due to eqs. (4) and (15) the other three relevant probabilities, $P(\nu_\beta \rightarrow \nu_\beta) = |A(\nu_\beta \rightarrow \nu_\beta)|^2$ and $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)}) = |A(\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)})|^2$, can all be expressed in terms of $P(\nu_\alpha \rightarrow \nu_\alpha)$. Using eqs. (12), (14), and (19) - (21) we get from eqs. (13a):

$$P(\nu_\alpha \rightarrow \nu_\alpha; t, t_0) = \bar{P}(\nu_\alpha \rightarrow \nu_\alpha) + \sum_{r=1}^4 P_r^{osc}(t, t_0), \quad (22)$$

where

$$\bar{P}(\nu_\alpha \rightarrow \nu_\alpha) = \frac{1}{2} + \left(\frac{1}{2} - P'\right) \cos 2\theta_m(t_0) \cos 2\theta_m(t), \quad (23)$$

is the average probability and

$$P_1^{osc}(t, t_0) = \sqrt{P'(1 - P')} \cos 2\theta_m(t_0) \sin 2\theta_m(t) \cos(\Phi_{12} + \Phi_{22}), \quad (24a)$$

$$P_2^{osc}(t, t_0) = -\sqrt{P'(1 - P')} \sin 2\theta_m(t_0) \cos 2\theta_m(t) \cos(\Phi_{12} - \Phi_{22}), \quad (24b)$$

$$P_3^{osc}(t, t_0) = -\frac{1}{2} P' \sin 2\theta_m(t_0) \sin 2\theta_m(t) (\cos 2\Phi_{12} + \cos 2\Phi_{22}), \quad (24c)$$

$$P_4^{osc}(t, t_0) = \frac{1}{2} \sin 2\theta_m(t_0) \sin 2\theta_m(t) \cos 2\Phi_{22}, \quad (24d)$$

are oscillating terms. The functions $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$ are responsible for the oscillatory dependence of the four probabilities $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\alpha(\beta)})$ and $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)})$ on the neutrino energy E and/or on the parameters characterizing the transitions being studied.

The result expressed by eqs. (22) - (24d) was derived for the probability of MSW transitions (see eq. (6)) of solar neutrinos in the Sun in ref. [19], utilizing the exact solutions of the system (1) found in the case of electron number density $N_e(t)$ changing exponentially along

the neutrino trajectory ⁹. Explicit analytic expressions for the “jump” probability P' as well as for the phase functions $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$ in the indicated case were also derived [13,19] (see also [26]). This permitted to perform a detailed study of the magnitude and the behavior under various averagings of the oscillating terms present in the solar neutrino MSW transition probability (for details see refs. [19,27]) ¹⁰.

For adiabatic transitions one has $P' \cong 0$ and consequently $P_{1,2,3}^{osc}(t, t_0) \cong 0$, while $P_4^{osc}(t, t_0)$ can be non-negligible. Thus, $P_{1,2,3}^{osc}(t, t_0)$ can be identified as nonadiabatic oscillating terms, while for $P' = 0$, $P_4^{osc}(t, t_0)$ should coincide with the oscillating term in the case of adiabatic transitions. The expression for the latter in terms of the functions $\epsilon(t)$ and $\epsilon'(t)$ can be easily derived from (1). It has the form of the term $P_4^{osc}(t, t_0)$, eq. (24d), which allows to determine $\Phi_{22}(t, t_0)$ when $P' \cong 0$:

$$\Phi_{22}^{AD}(t, t_0) = \int_{t_0}^t \sqrt{\epsilon^2(t') + \epsilon'^2(t')} dt'. \quad (25)$$

In the majority of cases explicit expressions for $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$ do not exist. Nevertheless, eqs. (22) - (24d) permit to determine, in particular, some of the conditions under which the oscillating terms give negligible contributions in the probabilities $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\alpha(\beta)})$ and $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)})$; they also allow one to obtain upper limits on these contributions when the latter are expected to be non-negligible. This can be done if the probability P' and the values of $\theta_m(t_0)$ and $\theta_m(t)$, i.e., the values of the $\epsilon(t)$ and $\epsilon'(t)$ in the initial and final points of neutrino trajectory, are known. In this case the amplitude of the oscillations described by each of the terms given in eqs. (24a) - (24d) can be calculated.

In many cases of physical interest one can use the Landau-Zener expression for the “jump” probability [12,16] (see also [15]):

$$P'_{LZ} = e^{-2\pi n_0}, \quad (26)$$

where

$$4n_0 = 4n(t = t_{res}) = 2 \frac{\epsilon'^2(t = t_{res})}{|\dot{\epsilon}(t = t_{res})|}, \quad (27)$$

⁹There are several misprints in ref. [19] (a list was given in ref. [27]). The three most relevant are: i) the overall minus sign in the right-hand side of eq. (40) should be canceled, ii) $\cos 2\theta_m(t_0)$ in eq. (41) should read $\sin 2\theta_m(t_0)$, and iii) a π should be added in the right-hand side of the relation between the $\Phi_{12}(t, t_0)$ and the phase of the $\nu_1^m(t_0) \rightarrow \nu_2^m(t)$ transition amplitude, given in footnote 9 ($\Phi_{12}(t, t_0)$ defined in [19] is expressed in terms of $\arg(-A_{12}^m)$ rather than of $\arg(A_{12}^m)$). Note that the probability amplitudes defined in [19] differ from those considered here by the phase factor $\exp(i \int_{t_0}^t \epsilon(t') dt')$, which is also reflected in the relation between $\Phi_{12}(t, t_0)$ and $\arg(A_{12}^m)$ derived in [19].

¹⁰To our knowledge, the case of MSW transitions of solar neutrinos in the Sun (exponentially varying density) is the only nontrivial and physically relevant one for which explicit analytic expressions for the phase functions $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$ were obtained so far.

is the adiabaticity parameter, which is just the value of the adiabaticity function

$$4n(t) = \frac{E_2^m - E_1^m}{2 |\dot{\theta}_m(t)|} = 2 \frac{(\epsilon^2(t) + \epsilon'^2(t))^{\frac{3}{2}}}{|\epsilon(t)\dot{\epsilon}'(t) - \dot{\epsilon}(t)\epsilon'(t)|} \quad (28)$$

at the resonance point. In order for a given type of transition to be adiabatic, the inequality $4n(t) \gg 1$ should be fulfilled at each point of the neutrino trajectory. In certain specific cases, such as MSW transitions of solar neutrinos in the Sun, the adiabaticity condition $4n(t) \gg 1$ is always satisfied if it is valid at the resonance point [2], i.e., if the inequality $4n_0 \gg 1$ holds.

The Landau-Zener expression for P' , eq. (26), was derived assuming that $\epsilon(t)$ decreases linearly with time (distance) while $\epsilon'(t)$ is constant, on the neutrino trajectory. The limits of its applicability for describing the matter-enhanced neutrino transitions in a medium are well-known [9,13,18]. It reproduces the “jump” probability rather accurately if the resonance region of the transition ¹¹ is sufficiently narrow, so that $\epsilon'(t)$ does not change substantially and the change of $\epsilon(t)$ in it can be well approximated by a linear function. Expression (26) cannot be used for description of nonadiabatic transitions when the point of neutrino production is located in the resonance region.

In certain specific cases the analytic expression for the “jump” probability, derived for matter (electron, neutron number) density changing exponentially along the neutrino trajectory [13], P'_{exp} , provides a more accurate description of the matter-enhanced neutrino transitions in a medium than P'_{LZ} . For MSW neutrino transitions, eq. (6), the exponential density result for P' reads :

$$P'_{exp} = \frac{e^{-2\pi n_0(1-\tan^2\theta)} - e^{-2\pi n_0(\tan^{-2}\theta-\tan^2\theta)}}{1 - e^{-2\pi n_0(\tan^{-2}\theta-\tan^2\theta)}} = \frac{e^{-2\pi r_0 \frac{\Delta m^2}{2E} \sin^2\theta} - e^{-2\pi r_0 \frac{\Delta m^2}{2E}}}{1 - e^{-2\pi r_0 \frac{\Delta m^2}{2E}}} . \quad (29)$$

Here $r_0 = |N(t = t_{res})/\dot{N}(t = t_{res})|$, $\dot{N}(t) = \frac{d}{dt}N(t)$ and $N(t) = N_e(t)$, i.e., r_0 is the scale-height of the change of $N_e(t)$ in the case of interest. The analogous expression for the “jump” probability corresponding to RSFP in a constant magnetic field, $P'_{exp}(RSFP)$, can formally be obtained from eq. (29) by replacing Δm^2 and θ by $\Delta m^2/\cos 2\delta$ and δ respectively, where $\tan 2\delta = 2\mu_\nu B_\perp/(\Delta m^2/2E)$, and by choosing the appropriate $N(t)$ in the expression for r_0 (in the case of eq. (7), for example, $N(t) = N_e(t) - N_n(t)$).

Unlike expression (26), the one given by eq. (29) describes correctly, for instance, the (strongly) nonadiabatic MSW transitions of solar neutrinos in the Sun for values of $\sin^2 2\theta \geq (0.2 - 0.3)$ [9,13,18]. In the case of RSFP in a magnetic field which varies along the neutrino trajectory, one should use the value of the field at the resonance point, $B_\perp(t = t_{res})$, in the expression for $P'_{exp}(RSFP)$, i.e., in the definition of $\tan 2\delta$ given above. To our knowledge, the accuracy of the descriptions of the RSFP of solar and supernova neutrinos, based on the “jump” probabilities P'_{LZ} and $P'_{exp}(RSFP)$, has never been thoroughly tested for magnetic fields which change along the neutrino trajectory.

¹¹This is the region of the neutrino trajectory around the resonance point, where $\sin^2 2\theta_m(t) \geq 1/2$.

Let us note that independently of the value of P' , the contribution of the oscillating terms $P_{1,2,3,4}^{osc}(t, t_0)$ in the probabilities $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\alpha(\beta)})$ and $P(\nu_{\alpha(\beta)} \rightarrow \nu_{\beta(\alpha)})$ will be suppressed, as it follows from eqs. (24a) - (24d), if, for instance, $\sin 2\theta(t_0) \cong 0$ (e.g., $\theta(t_0) \cong \pi/2$) and $\sin 2\theta(t) \cong 0$ (e.g., $\theta(t) \cong 0$). In the case of adiabatic transitions we have $P' \cong 0$ and $P_{1,2,3}^{osc}(t, t_0) \cong 0$, while the oscillating term $P_4^{osc}(t, t_0)$ is suppressed provided $\sin 2\theta(t_0) \cong 0$ or $\sin 2\theta(t) \cong 0$.

It is quite well-known (see, e.g., [18]) that in the limits of sufficiently large and sufficiently small $\Delta m^2/(2E)$ in the MSW case, the solar neutrinos oscillate in the Sun as in vacuum. It is instructive to see how expression (22) for $P(\nu_e \rightarrow \nu_e)$ reduces to the vacuum oscillation one in these two limits. For the solar neutrino MSW transitions in the Sun one has $\theta_m(t) = \theta$ (see eqs. (6) and (9)) since $N_e(t) = 0$ at the surface of the Sun. If $\Delta m^2/(2E)$ is sufficiently large, we have $4n_0 \gg 1$ and $N_e^{res} = \Delta m^2 \cos 2\theta/(2E\sqrt{2}G_F) \gg N_e(t_0) \geq N_e(t')$, $t_0 \leq t' \leq t$, where $N_e(t_0)$ is the electron number density in the point of ν_e production in the Sun. Consequently, as it follows from eqs. (6), (9), (25) and (26) (or (29)), $\theta_m(t_0) \cong 0$, $P' \cong 0$, and $\Phi_{22} \cong \Delta m^2(t - t_0)/(2E)$. Using the above equalities and eqs. (22) - (24d) it is easy to convince oneself that $P(\nu_e \rightarrow \nu_e)$ indeed reduces to the vacuum oscillation probability. We get the same result when $\Delta m^2/(2E)$ is sufficiently small, so that $\cos 2\theta_m(t_0) \cong -1$ ($P_{2,3,4}^{osc} \cong 0$) and $4n_0 \ll 1$, i.e., the solar neutrinos undergo extremely nonadiabatic transitions. In this case eq. (29) implies ¹² $P' \cong \cos^2 \theta$ and we have [19] $\Phi_{12} + \Phi_{22} \cong \pi + \Delta m^2(t - t_0)/(2E)$. Therefore $\bar{P}(\nu_e \rightarrow \nu_e)$ and P_1^{osc} reduce to the average probability and the oscillating term in the solar neutrino vacuum oscillation probability (see, e.g., [24]): $\bar{P}(\nu_e \rightarrow \nu_e) \cong 1 - 1/2 \sin^2 2\theta$ and $P_1^{osc} \cong 1/2 \sin^2 2\theta \cos(\Delta m^2(t - t_0)/(2E))$. Note that the nonadiabatic oscillating term P_1^{osc} converges to the vacuum oscillating term in this case.

We would like to conclude this Section with the following remark. In certain cases of neutrino transitions in a medium one is faced with the problem of calculating the real part of the product of the two amplitudes $A^*(\nu_\alpha \rightarrow \nu_\alpha)$ and $A(\nu_\alpha \rightarrow \nu_\beta)$, arising as an interference term in the corresponding transition probability ¹³. Using eqs. (13a), (13b) and (19) - (21) it is not difficult to find the expression for the indicated product of amplitudes in terms of P' , $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$:

$$\begin{aligned} Re [A^*(\nu_\alpha \rightarrow \nu_\alpha)A(\nu_\alpha \rightarrow \nu_\beta)] = & - \left(\frac{1}{2} - P' \right) \cos 2\theta_m(t_0) \sin 2\theta_m(t) + \sqrt{P'(1 - P')} \cos 2\theta_m(t_0) \\ & \times \cos 2\theta_m(t) \cos(\Phi_{12} + \Phi_{22}) + \sqrt{P'(1 - P')} \sin 2\theta_m(t_0) \sin 2\theta_m(t) \cos(\Phi_{12} - \Phi_{22}) \\ & - \frac{1}{2} P' \sin 2\theta_m(t_0) \cos 2\theta_m(t) (\cos 2\Phi_{12} + \cos 2\Phi_{22}) + \frac{1}{2} \sin 2\theta_m(t_0) \cos 2\theta_m(t) \cos 2\Phi_{22}. \quad (30) \end{aligned}$$

Obviously, the terms which depend on the phase functions $\Phi_{12}(t, t_0)$ and/or $\Phi_{22}(t, t_0)$ are oscillating terms. Note that the factors multiplying a given oscillating function ($\cos(\Phi_{12} +$

¹²The Landau-Zener expression for P' , eq. (26), leads to an incorrect result since it is not valid, in particular, for relatively small $\Delta m^2/(2E)$ [9,13,18].

¹³This term appears, for example, in the probability of combined MSW transitions and long wave length vacuum oscillations of solar neutrinos in the case of three flavour-neutrino mixing [28].

Φ_{22}), etc.) in the expressions for $P(\nu_\alpha \rightarrow \nu_\alpha; t, t_0)$ and $\text{Re } [A^*(\nu_\alpha \rightarrow \nu_\alpha)A(\nu_\alpha \rightarrow \nu_\beta)]$, i.e., the oscillation amplitudes, are different. The result (30) was used in the recent study [28] of the hybrid MSW + vacuum oscillation solution of the solar neutrino problem. It should be clear that using eqs. (13a) - (13d), (19) and (20) we can express any product of two of the amplitudes $A^{(*)}(\nu_\alpha \rightarrow \nu_{\alpha(\beta)})$ and $A^{(*)}(\nu_\beta \rightarrow \nu_{\beta(\alpha)})$ in terms of P' , Φ_{12} , Φ_{22} , $\theta_m(t_0)$ and $\theta_m(t)$.

3. MSW Transitions of Solar Neutrinos in the Sun and the Hydrogen Atom

In the present Section we demonstrate that the second order differential equation for the amplitude $A(\nu_e \rightarrow \nu_e)$, which describes the MSW transitions of solar neutrinos in the Sun, coincides in form in the case of $N_e(t)$ changing exponentially along the neutrino path, with the Schroedinger equation for the radial part of the non-relativistic wave function of the hydrogen atom and we comment briefly on this interesting coincidence.

Assuming that i) $\nu_\alpha \equiv \nu_e$, $\nu_\beta \equiv \nu_{\mu(\tau)}$, ii) the initial conditions (A) are valid ($A_e(t, t_0) = A(\nu_e \rightarrow \nu_e)$), and substituting $\epsilon(t)$ and $\epsilon'(t)$ in eqs. (16) and (17) with their expressions given in eq. (6), we obtain from (17) a second order differential equation for the amplitude $A(\nu_e \rightarrow \nu_e)$, describing the MSW $\nu_e \rightarrow \nu_{\mu(\tau)}$ transitions. According to the contemporary solar models [17], $N_e(t)$ changes approximately exponentially along the trajectory of a solar neutrino moving radially towards the surface of the Sun:

$$N_e(t) = N_e(t_0) \exp \left\{ -\frac{t-t_0}{r_0} \right\}, \quad (31)$$

where $(t - t_0) \cong d$ is the distance traveled by the neutrino in the Sun, $N_e(t_0)$ and r_0 have been defined earlier, $r_0 \sim 0.1R_\odot$, $R_\odot = 6.96 \times 10^5 \text{ km}$ being the solar radius. Introducing the dimensionless variable

$$Z = ir_0\sqrt{2}G_F N_e(t_0) e^{-\frac{t-t_0}{r_0}}, \quad Z_0 = Z(t = t_0), \quad (32)$$

and making the substitution

$$A_e(t, t_0) \equiv A(\nu_e \rightarrow \nu_e) = (Z/Z_0)^{c-a} e^{-(Z-Z_0)+i \int_{t_0}^t \epsilon(t') dt'} A'_e(t, t_0), \quad (33)$$

we find that the amplitude $A'_e(t, t_0)$ satisfies [13,14,19] the confluent hypergeometric equation [29]:

$$\left\{ Z \frac{d^2}{dZ^2} + (c - Z) \frac{d}{dZ} - a \right\} A'_e(t, t_0) = 0, \quad (34)$$

where [19]

$$a = 1 + ir_0 \frac{\Delta m^2}{2E} \sin^2 \theta, \quad c = 1 + ir_0 \frac{\Delta m^2}{2E}. \quad (35)$$

The equation (34) coincides in form with the Schroedinger (energy eigenvalue) equation obeyed by the radial part, $\psi_{kl}(r)$, of the non-relativistic wave function of the hydrogen atom [30], $\Psi(\vec{r}) = \frac{1}{r} \psi_{kl}(r) Y_{lm}(\theta', \phi')$, where r , θ' and ϕ' are the spherical coordinates of the electron in the proton's rest frame, l and m are the orbital momentum quantum numbers ($m = -l, \dots, l$), k is the quantum number labeling (together with l) the electron energy ¹⁴,

¹⁴The principal quantum number is equal to $(k + l)$ [30].

E_{kl} ($E_{kl} < 0$), and $Y_{lm}(\theta', \phi')$ are the spherical harmonics. To be more precise, the function $\psi'_{kl}(Z) = Z^{-c/2} e^{Z/2} \psi_{kl}(r)$ satisfies equation (34), where the variable Z and the parameters a and c are in this case related to the physical quantities characterizing the hydrogen atom:

$$Z = 2 \frac{r}{a_0} \sqrt{-E_{kl}/E_I}, \quad a \equiv a_{kl} = l + 1 - \sqrt{-E_I/E_{kl}}, \quad c \equiv c_l = 2(l + 1), \quad (36)$$

where $a_0 = \hbar/(m_e e^2)$ is the Bohr radius and $E_I = m_e e^4/(2\hbar^2) \cong 13.6$ eV is the ionization energy of the hydrogen atom. It is remarkable that the behavior of such different physical systems as solar neutrinos undergoing MSW transitions in the Sun and the non-relativistic hydrogen atom are governed by one and the same differential equation.

The properties of the linearly independent solutions of eq. (34), i.e., of the confluent hypergeometric functions, $\Phi(a, c; Z)$, as well as their asymptotic series expansions, are well-known [29]. Any solution of (34) can be expressed as a linear combination of two linearly independent solutions of (34), $\Phi(a, c; Z)$ and $Z^{1-c} \Phi(a-c+1, 2-c; Z)$, which are distinguished from other sets of linearly independent confluent hypergeometric functions by their behavior when $Z \rightarrow 0$: $\Phi(a', c'; Z = 0) = 1$, $a', c' \neq 0, -1, -2, \dots$, a' and c' being arbitrary parameters. Explicit expressions for the probability amplitudes $A(\nu_e \rightarrow \nu_e)$ and $A(\nu_e \rightarrow \nu_{\mu(\tau)})$ in terms of the functions $\Phi(a, c; Z)$ and $\Phi(a - c + 1, 2 - c; Z)$ were derived in [19,26]. In the case of MSW transitions of solar neutrinos ($N_e(t) = 0$) these expressions have an especially simple form: they are given by the corresponding vacuum oscillation amplitudes “distorted” by the values of the functions $\Phi(a', c'; Z)$ in the initial point of the neutrino trajectory,

$$A(\nu_e \rightarrow \nu_{\mu(\tau)}) = \frac{1}{2} \sin 2\theta \left\{ \Phi(a - c, 2 - c; Z_0) - e^{i(t-t_0)\frac{\Delta m^2}{2E}} \Phi(a - 1, c; Z_0) \right\}, \quad (37)$$

etc., where Z_0 , a and c are defined in eqs. (32) and (35). In the limit $|Z_0| \rightarrow 0$, which corresponds to zero electron number density, expression (37) reduces to the one for oscillations in vacuum.

It is well-known that the requirement of a correct asymptotic behavior of the wave function $\psi_{kl}(r)$ at large r leads to the quantization condition for the energy of the electron, E_{kl} , in the hydrogen atom [30]: $E_{kl} = -E_I/(k + l)^2$, $(k + l) = 1, 2, \dots$ ($l = 0, 1, 2, \dots, (k + l) - 1$). Technically, the condition is derived by using the asymptotic series expansion of the confluent hypergeometric functions in powers of the argument Z [29] (one has $Z \rightarrow \infty$ when $r \rightarrow \infty$, see eq. (36)). The same asymptotic series expansion in the case of the solutions describing the MSW transitions of solar neutrinos in the Sun (we have $|Z_0| \gtrsim 520$ in this case [19]) permitted to obtain expression (29) for the “jump” probability P' [13], as well as explicit expressions for the phase functions $\Phi_{12}(t, t_0)$ and $\Phi_{22}(t, t_0)$ [19].

4. Conclusions

We have derived an exact universal analytic expression for the probability of two-neutrino matter-enhanced transitions in a medium (MSW, RSFP, induced by neutrino FCNC interaction, etc.). The probability is expressed in terms of three real functions of the parameters characterizing the neutrino transitions: the “jump” probability P' and two phases (angles) Φ_{12} and Φ_{22} . The latter are responsible for the oscillatory dependence of the probability on the parameters of the problem being investigated (neutrino energy, matter density, magnetic field strength, etc.). Although in the majority of cases of physical interest the two phase

functions Φ_{12} and Φ_{22} are not known, the amplitudes of the oscillations due to Φ_{12} and Φ_{22} are functions only of the “jump” probability and the values of the neutrino mixing angle in matter at the initial and final points of the neutrino trajectory, i.e., of quantities which are typically estimated or calculated in the studies of the neutrino matter-enhanced transitions. Our results can be used, in particular, in the investigations of matter-enhanced transitions of solar and supernova neutrinos.

We have noted also that the second order differential equation for the MSW probability amplitude of solar ν_e survival in the Sun coincides in form, for solar matter (electron number) density changing exponentially along the neutrino trajectory, with the Schrödinger equation for the radial part of the non-relativistic wave function of the hydrogen atom. The equation is of a confluent hypergeometric type and the asymptotic series expansion of its solutions plays important role in the description of both physical systems: it permits to obtain, for instance, the quantization condition for the electron energy levels in the hydrogen atom and the analog of the Landau-Zener “jump” probability for exponentially varying density in the case of MSW transitions of solar neutrinos in the Sun.

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